Project: Pythagorean Theorem
MATH 3363
Spring, 2011

You will investigate five different proofs of the Pythagorean Theorem. Based on your investigations you are to compare and contrast the proofs presented.

Because of the amount of work, students should work in groups of two or three. Divide the work and then devise a way to put it all together.

Each group must submit the following:

- A file with all relevant sketches. Files submitted must be clearly labeled so that I can tell which sketch goes with which proof. (For example, "sketch#l" is not an acceptable file name.}
- A word-processed copy of any required proofs (this may depend on the particular proofs you are investigating}. Note: the sketch you submit is NOT considered a proof- you will need to provide a general argument for a proof based on the sketch you've constructed.
- A word-processed report of your synthesis of the activities. [That is, I want you to compare/contrast the five proofs.] Your synthesis must include an introductory paragraph, a body (the main part of your paper}, and a concluding paragraph. It must be written in Standard English and conform to standard usage. There is to be ONE report per group- thus, it must be a group effort. Care should be taken in the editing that it does not read as if different people wrote the paper. This does not mean it should be written by one person only, but if there are multiple contributors, there should be someone in the group acting as an editor.

Due date: May 6th – 12 noon

Behold!

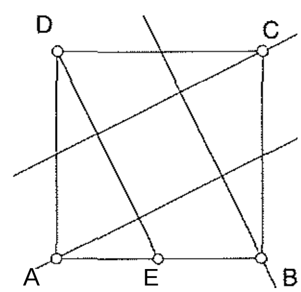
Name(s): ____________________________

The title of this activity comes from the text of the twelfth-century Hindu scholar, Bhaskara. In fact, "Behold!" was the *only* text that accompanied a figure demonstrating the Pythagorean theorem. Bhaskara must have felt the figure spoke for itself! In this activity, you'll construct this figure. Perhaps it does speak for itself, but you can gain deeper understanding by constructing the figure and working out a proof similar to the proof outlined in the activity The Tilted Square Proof. Incidentally, this figure is also found in an ancient Chinese text, making it another candidate for being a proof known to Pythagoras.

The figure in the Chinese text shows a 3-4-5 right triangle, but the figure works for any right triangle.

Sketch and Investigate

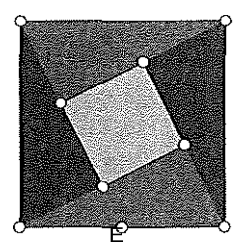
1. Construct a square *ABCD.*
2. From point D, construct a segment *DE* to side *AB.*



3. Construct a line parallel to *DE* through point *B* and lines perpendicular to *DE* through points *A* and C.
4. You should have a small, tilted square inside your original square. Construct its vertices at the points of intersection of th2:_ lines, then hide the lines and *DE.*

You can do all of this with the polygon tool before hiding the lines & DE

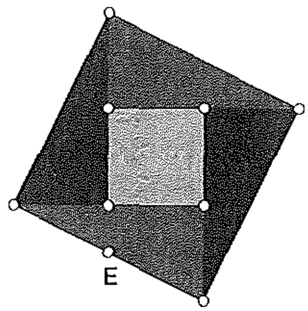
5. Construct the sides of the tilted square, then construct interiors of the right triangles surrounding it, as shown.

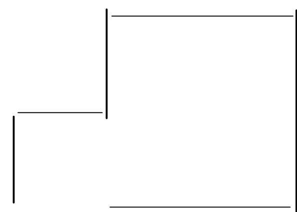


6. Drag point *E* and observe what happens to the right triangles and the tilted square.

Q1 Write a paragraph about what you observe: Do the right triangles stay right triangles? Does the square stay a square? Can you make the interior square fill the figure? If so, what kind of triangles do you get in this case? Can you make the interior square disappear? What kind of triangles will do this?

7. Using this figure for a dissection demonstration can be tricky. First, tilt the original square so that the legs of the right triangles are horizontal and vertical. The illustration below shows an outline into which the pieces can be fit to demonsh·ate that $c^2 = a^2 + b^2$. *Note:* Use the **Translator** tool to get movable pieces, and try to form the shape below right. Your pieces may overlap the dashed line.""





Prove

This figure can be used to create an algebraic proof similar to the one you might have done in the activity The Tilted Square Proof. A possible plan follows. Read it if you want, or try it on your own.

Q2 Write an expression for the area of the whole square in terms of c.

Q3 Write an expression for the sum of the areas of the four right triangles in terms of *a* and *b*.

Q4 The tricky part is writing an expression for the area of the tilted square. (And if you want to be vety thorough, you should prove it really is a square.) The length of a side of this square can be written in terms of *a* and *b*. Write an expression for the area of the tilted square in terms of *a* and *b*.

QS You should now be able to write an equation involving *a, b,* and c. You're on your own from here.

Leonardo da Vinci's Proof

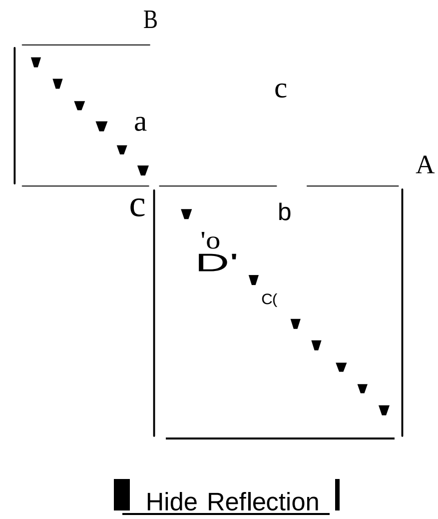
Name(s): ___________________________

Leonardo da Vinci (1452-1519) was a great Italian painter, engineer, and inventor during the Renaissance. He is famous for, among other things, painting the Mona Lisa. He is also credited with the following proof of the Pythagorean theorem.

Sketch and Investigate

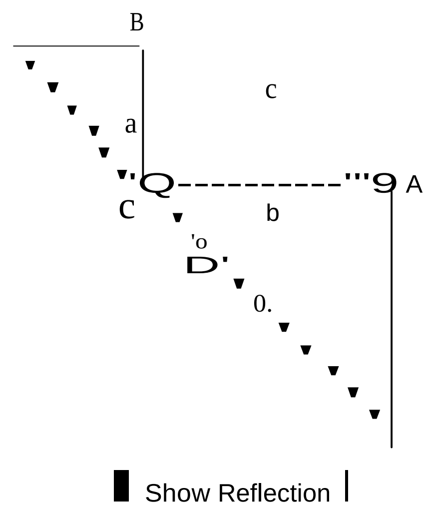
In this figure, you don't have to construct the square on the hypotenuse.

1. Construct a right triangle and squares on the legs.
2. Connect corners of the squares to construct a second right triangle congruent to the original.
3. Construct a segment through the center of this figure, connecting far corners of the squares and passing through C.
4. Construct the midpoint, D, of this segment.

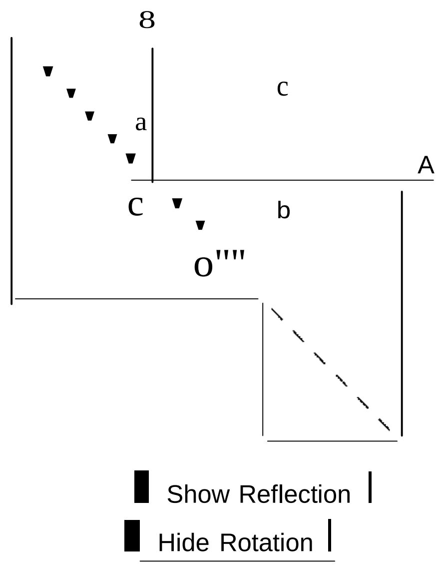


The Action Button submenu is in the Edit menu.

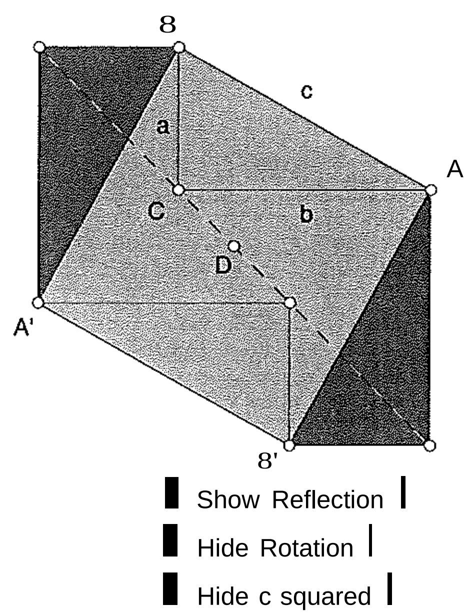
5. This segment divides the figure into mirror image halves. Select all the segments and points on one side of the center line and create a Hide/Show action button. Change its label to read *Hide Reflection.*
6. Press *Hide Reflection.* You should now see half the figure.



7. Mark *D* as center and rotate the entire figure (not the action button) 180° around D.
8. Select all the objects making up the rotated half of this figure and create a Hide/Show action button. Relabel the button to read *Hide Rotation,* but don't hide the rotated half *yet.*



9. Construct *A* Band *B'A.* Do you see *c* squared?
10. Construct the polygon interior of *BA' B'A* and of the two h'iangles adjacent to it.
11. Select *A'B, B'A,* and the three polygon interiors and create a Hide/Show action button. Name this *Hide c squared.*



Going through this construction *may* give you a good idea of how da Vinci's proof goes.

12. Press each of the Hide buttons, then play tlu·ough the buttons in this sequence: *Show Reflection, Show Rotation, Hide Reflection, Show c squared.*

You should see the transformation from two right triangles with squares on the legs into two identical right triangles with a square on their hypotenuses.

Leonardo da Vinci's Proof (continued)

-Qf Explant to a classntale or ntake a presentation to ilie class to da Vffiei'*pwGi-o£* the Pythagorean theerem.

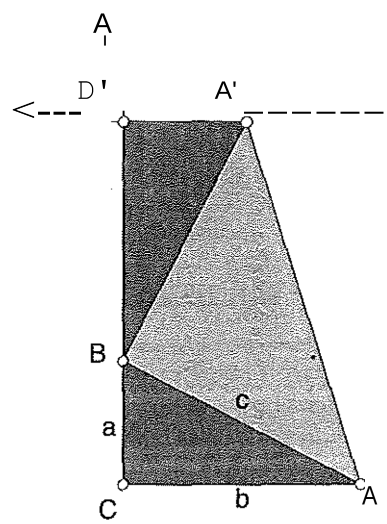
Q2 Da Vinci's is another of those elegant proofs where the figure tells pretty much the whole story. Write a paragraph that explains why the two hexagons have equal areas and how these equal hexagons prove the Pythagorean theorem.

Presidential Pythagoras

Name(s): ____________________

James A Garfield discovered a proof of the Pythagorean theorem in 1876, a few years before he became president of the United States. An interest in mathematics may not have been a prerequisite for the presidency, but it must have been common at the time. One of Garfield's predecessors, Abraham Lincoln, credited Euclid's *Elements* as being one of the books most influential to his career as a laVIryer and politician, saying he learned from it how to think logically. Garfield's Pythagorean theorem proof is illustrated with a relatively simple figure: a trapezoid.

Sketch and Investigate



1. Construct a right triangle *ABC* and label it as shown.
2. Mark point *B* as center and rotate side *c* and point *A* by 90°.
3. Connect points *A* and *A'* and construct a line through point *A'*, parallel to side *b*.
4. Use the Ray tool to extend side *CB* and construct the point of intersection, D, of this ray and the line tlu-ough *A'*.
5. Hide the ray and the line and replace them with segments *BD* and *DA'*.
6. Construct polygon interiors for the three right triangles.

Q1 What kind of figure is quadrilateral *ACDA'*? How do you know? Drag points *A*, *B*, and C. Does the figure remain this type of special quadrilateral?

Q2 What can you say about the triangles into which *ACDA'* is divided? Do they maintain these properties when you drag different parts of your sketch?

The point here is to use only side lengths and **Calculate** from the Measure menu to calculate areas. Don't actually measure areas until you're ready to confirm your calculations.

7. Measure sides *a*, *b*, and c. Now use just these measures to calculate the areas of the three triangles and their sum.
8. Now use the area formula for a trapezoid to calculate the area of *ACDA'* using just the side lengths. (What's the height of trapezoid *ACDA'*?) Construct the polygon interior of the entire figure and confirm your calculations were done correctly.

Presidential Pythagoras (continued)

Prove

In steps 7 and 8, you calculated the area of the trapezoid in two different ways. Garfield used these two different ways of finding the area to prove the Pythagorean theorem. Can you do it too? Write two different expressions for the same area in terms of $a, b,$ and c. Set these expressions equal and do the necessary algebra to arrive at the Pythagorean theorem.

EXplore More

L-t!ok at the figure in the aetivity The Tilted Square Proof. How is Garfield's figure telated to this one? See if you ealt tf!lfts£orm (that's.a !ti:tttJ Ga:rfield's figwe into tlre 'filled Square figure. Compare the algebra in*v* oh ed in the *t?.•w* proofand write a paragraph about hew tbe proofs are ₁elatee.

A Dynamic Proof

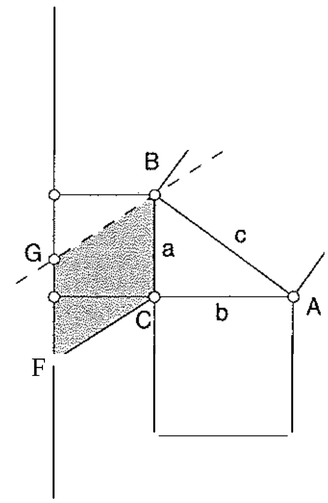
Name(s): ____________________________

Unlike dissection proofs, in which you move pieces to different locations in your figure without changing them, in this construction you'll transform your squares, without changing their areas, to create congruent shapes. To do this, you'll actually construct parallelograms on the sides of your right triangle:

Sketch and Investigate

1. Construct a right triangle and squares on the sides.
2. Hide the far side of the square on side *a* and replace it with a line.
3. Construct *CF*, where *F* is a point on the line.
4. Construct a line through *B*, parallel to *CF*.
5. Construct the interior of parallelogram *BCFG* and hide the lines.

throug : construct a parallelogram on s1de *a*

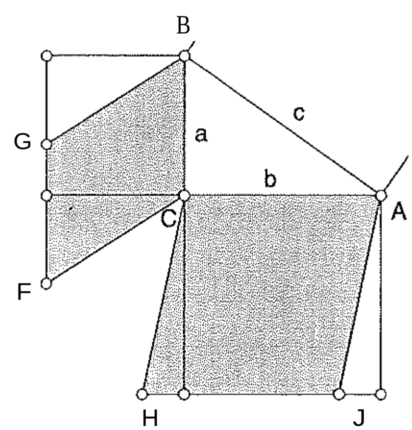


This transformation is called a shear. A shear translates every point in a figure in a direction parallel to a given line by a distance proportional to the point's distance from the line. Shearing a figure preserves its area.

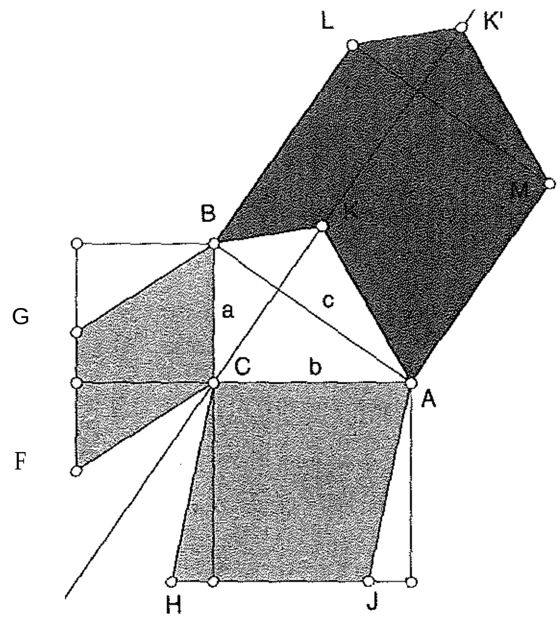
Drag point *F* to experiment with the parallelogram's behavior. The key to this demonstration is that the parallelogram's area is always the same as the square whose side it shares. Can you see why?

6. Construct a parallelogram on side *b* by the same method you used in steps 2 through 5 above, starting with segment *CH* so that *H* is a point that can be dragged.

Explain why this is true.



You should now have parallelograms on sides *a* and *b* that can be dragged back and forth by one vertex without changing their areas.



7. Construct a line tlu·ough point C, perpendicular to side c.

8. Construct *BK* and *AK*, where *K* is a point on this line.

9. Mark BL as a vector and translate *BK*, point *K*, and *AK* by this marked vector.

Note that this concave hexagon consists of two paraHelograms, *BLK'K* and *AKK'M*.

r 10. Construct the polygon interior *BLK'MAK.*

11. Drag point *K* to make sure polygon *BLK'MAK* behaves correctly: Point *K'* should move when you drag *K* so that *BLK'K* and *AKK'M* remain parallelograms.

If everything is working properly, you're ready to manipulate your sketch to give a dynamic proof of the Pythagorean theorem.

In your sketch, you may need to drag a different point, depending on how you constructed the parallelogram.

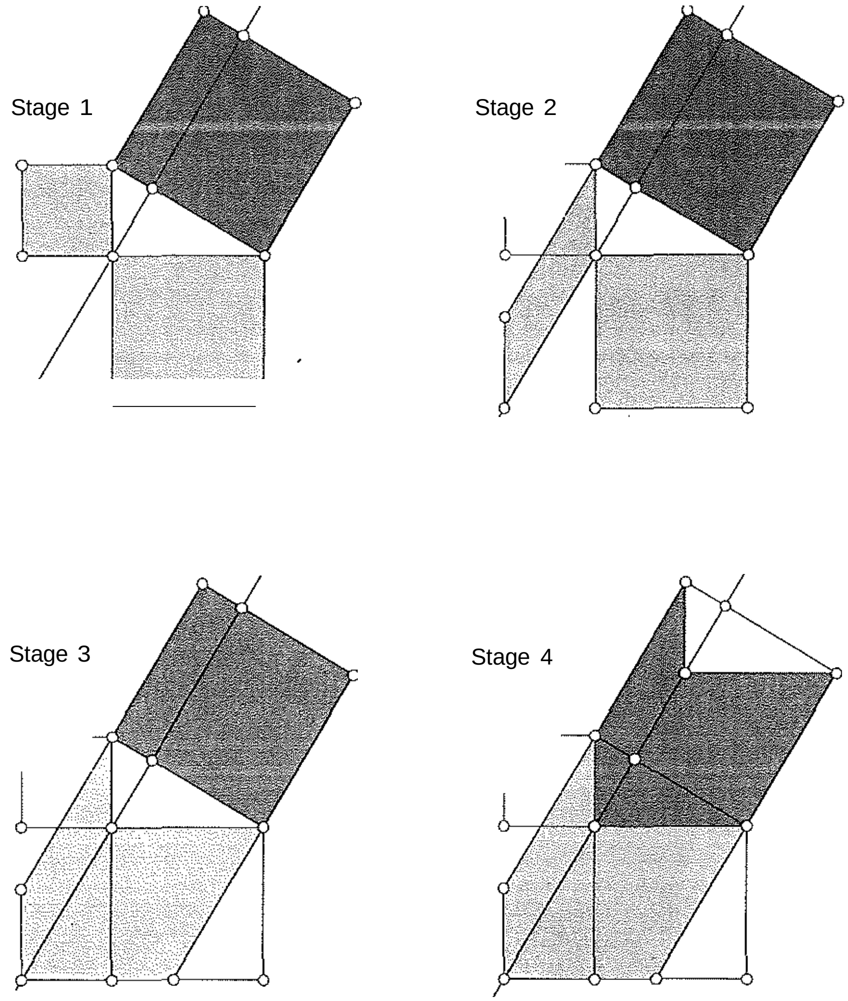
r>12. Drag point *F* to shear the parallelogram back and forth. Note that the parallelogram can fill the square. Note too that the parallelogram's area doesn't change. (You can measure to confirm this.) Finish the shear by dragging *F* to lie on the line through C and *K.*

Q1 Explain why the area of the parallelogram doesn't change.

13. Shear the parallelogram on the other leg of the h·iangle. Note that it too fills its square and that its area doesn't change. Drag until point *H* is on the line.

14. Drag point *K* to shear the parallelograms in the square on c. Note that these parallelograms can fill the square and that the area of the polygon (the sum of the areas of the parallelograms) doesn't change. Drag until point *K* coincides with point C.

Q2 The figures on the next page show the sequence you might have performed. How does this demonstrate the Pythagorean theorem?



Prove

The dynamic "proof' you perform by manipulating your sketch to h·ansform the squares into two congruent figures may seem proof enough of the Pythagorean theorem. People might consider the details of the proof superfluous compared to the convincing power of the dynamic demonstration. Yet if you were showing this proof to someone, you'd probably want to explain what's going on and why your sketch proves the theorem. And to make it a complete, or rigorous, proof, you'd want to supply the details. Start with a figure like that in stage 4 above. Add labels and write a complete proof. That is, show that each parallelogram is equal in area to a square (or part of a square) and show that the two pairs of parallelograms are congruent.